AN ESTIMATE OF FREE ENTROPY AND APPLICATIONS

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Abstract. We obtain an estimate of free entropy of generators in a type II_1-factor $\mathcal{M}$ which has a subfactor $\mathcal{N}$ of finite index with a subalgebra $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \subset \mathcal{N}$ where $\mathcal{P}_1 = \mathcal{R}_1' \cap \mathcal{P}$, $\mathcal{P}_2 = \mathcal{R}_2' \cap \mathcal{P}$ are diffuse, $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{P}$ are mutually commuting hyperfinite subfactors, and an abelian subalgebra $\mathcal{A} \subset \mathcal{N}$ such that the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N}, \tau)_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in a subcorrespondence ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M}, \tau)_{\mathcal{A}}$, generated by v vectors. The (modified) free entropy dimension of any generating set of $\mathcal{M}$ is $\leq 2r + 2v + 4$, where r is the integer part of the index. As a consequence, the interpolated free group subfactors of finite index do not have regular non-prime subfactors or regular diffuse hyperfinite subalgebras.

1. Introduction

D. Voiculescu defined ([Vo2], [Vo3]) the original concepts of free entropy and of (modified) free entropy dimension for m-tuples of self-adjoint non-commutative random variables. Very roughly, the free entropy $\chi((x_i)_{1\leq i\leq m})$ is a normalized limit of logarithms of volumes of sets of matricial microstates (that is, m-tuples of matrices whose non-commutative moments approximate those of $(x_i)_{1\leq i\leq m}$), while the modified free entropy dimension $\delta_0((x_i)_{1\leq i\leq m})$ is in some sense an asymptotic Minkowski dimension of the sets of matricial microstates. Then he proved ([Vo3]) that $\delta_0((x_i)_{1\leq i\leq m}) \leq 1$ if the von Neumann algebra $\{(x_i)_{1\leq i\leq m}\}''$ has a regular diffuse hyperfinite $*$-subalgebra (DHSA). Since the free group factors have generators with $\delta_0 > 1$, this implied in particular the absence of Cartan subalgebras in the free group factors, thus answering in the negative the longstanding open question of whether every separable II_1-factor arises from a measurable equivalence relation.

A. Connes introduced Kazhdan's property T from groups ([Ka]) to the von Neumann algebras context in [Co1] where he proved that if Γ is a countable discrete ICC group with property T, then the fundamental group of the von Neumann algebra of Γ is countable. This remarkable

2000 *Mathematics Subject Classification.* Primary 46Lxx; Secondary 47Lxx.

rigidity result produced the first examples (such as $\mathcal{L}(SL(3,\mathbb{Z}))$) of II_1-factors with fundamental group strictly smaller than $\mathbb{R}_+^\times$. A. Connes and V. Jones ([CoJo]) defined then property T for arbitrary von Neumann algebras in terms of correspondences ([Co2], [Co3]) and showed that the countable discrete ICC group Γ has property T if and only if $\mathcal{L}(\Gamma)$ has property T. Correspondences play the role of group representations and property T can be naturally defined by following this analogy. Thus, the space of (equivalence classes of) correspondences can be endowed with a topology through their coefficients and property T simply means that the trivial correspondence is isolated from the correspondences that do not contain it.

The free group factors $\mathcal{L}(\mathbb{F}_n)$, $2 \leq n < \infty$, were the first examples of prime (i.e., non-isomorphic to tensor products of) type II_1-factors with separable preduals. Their primeness (conjectured by S. Popa in [Po3]) was proved by L. Ge ([Ge2]) via an estimate of free entropy. Also with free entropy estimates and extending D. Voiculescu's result about the absence of Cartan subalgebras, L. Ge ([Ge1]) and K. Dykema ([Dy2]) showed that the free group factors do not have abelian subalgebras of multiplicity one and of finite multiplicity, respectively.

We consider subcorrespondences ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M},\tau)_{\mathcal{A}}$, where $\mathcal{N} \subset \mathcal{M}$ is an inclusion of type II_1-factors with finite Jones index (with integer part equal to r) and $\mathcal{P}$, $\mathcal{A}$ are von Neumann subalgebras of $\mathcal{N}$. We assume moreover that $\mathcal{A}$ is abelian and $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ where $\mathcal{P}_1 = \mathcal{R}_1' \cap \mathcal{P}$ and $\mathcal{P}_2 = \mathcal{R}_2' \cap \mathcal{P}$ are both diffuse and $\mathcal{R}_1$, $\mathcal{R}_2$ are mutually commuting hyperfinite subfactors. Then we prove (Theorem 4.1, using Lemma 3.1) that if the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in ${}_{\mathcal{P}}H_{\mathcal{A}}$ and if ${}_{\mathcal{P}}H_{\mathcal{A}}$ is spanned by v vectors, then the (modified) free entropy dimension of any generating set of $\mathcal{M}$ is $\leq 2r + 2v + 4$. For $\mathcal{M} = \mathcal{L}(\mathbb{F}_t)$ (an interpolated free group factor from [Dy1], [Ră]), the free entropy dimension estimate implies (Theorem 4.2) the absence of regular non-prime subfactors and of regular hyperfinite diffuse subalgebras (DHSA) in the subfactors $\mathcal{N} \subset \mathcal{L}(\mathbb{F}_t)$ $(1 < t \leq \infty)$ of finite Jones index. In particular, the interpolated free group subfactors of finite index are not crossed products of non-prime subfactors or hyperfinite diffuse subalgebras by properly outer actions of countable discrete groups (Corollary 4.2). We mention that the Haagerup approximation property ([Ha]), primeness, and absence of abelian subalgebras of finite multiplicity (and thus of Cartan subalgebras) are known to be preserved ([Şt1], [Şt2]) to the interpolated free group subfactors of finite index.

2. Notations

We shall use $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$, $\mathcal{A}$ etc. to denote (finite) von Neumann algebras. In particular, we use $\mathcal{M}$ for a type II_1-factor and $\mathcal{N}$ for a subfactor of $\mathcal{M}$. Let $\mathcal{P}$ be a finite von Neumann algebra, endowed with a normal faithful tracial state $\tau : \mathcal{P} \to \mathbb{C}$. For any projection $p \in \mathcal{P}$ we denote by $\mathcal{P}_p = p\mathcal{P}p$ the corresponding reduced von Neumann algebra of $\mathcal{P}$. Note that the functional $\tau_p : \mathcal{P}_p \to \mathbb{C}$, $\tau_p(y) = \frac{1}{\tau(p)}\tau(y)\ \forall y \in \mathcal{P}_p$ is a normal faithful tracial state on $\mathcal{P}_p$. The completion of $\mathcal{P}$ with respect to the 2-norm $||x||_2 = \tau(x^*x)^{\frac{1}{2}}\ \forall x \in \mathcal{P}$ is a Hilbert space, denoted $L^2(\mathcal{P}, \tau)$. If $\mathcal{N} \subset \mathcal{M}$ is an inclusion of type II_1-factors, then the Jones index $[\mathcal{M} : \mathcal{N}]$ is, by definition ([Jo]), the dimension $\dim_{\mathcal{N}} L^2(\mathcal{M}, \tau)$ of the left $\mathcal{N}$-module $L^2(\mathcal{M}, \tau)$. We mention that the dimension $\dim_{\mathcal{N}} H$ of an arbitrary left $\mathcal{N}$-module H was introduced by F. Murray and J. von Neumann ([MvN]) as the *coupling constant* of H. While $\dim_{\mathcal{N}} H$ assumes all possible values from $[0, \infty]$, the celebrated result of V. Jones ([Jo]) shows that necessarily $[\mathcal{M} : \mathcal{N}] \in \left\{4\cos^2 \frac{\pi}{n} : n \geq 2\right\} \cup [4, \infty]$. Many interesting von Neumann algebras arise from representations of discrete groups. For example, let Γ be a discrete group and denote by $(\delta_\gamma)_{\gamma\in\Gamma}$ the standard orthonormal basis in $l^2(\Gamma)$. If $\lambda : \Gamma \to \mathcal{B}(l^2(\Gamma))$, $\lambda_{\gamma_1}\delta_{\gamma_2} = \delta_{\gamma_1\gamma_2}\ \forall \gamma_1, \gamma_2 \in \Gamma$ denotes the left regular representation of Γ, then the (left) group von Neumann algebra is $\mathcal{L}(\Gamma) = \lambda_\Gamma''$. Moreover, it is easily seen that $\mathcal{L}(\Gamma)$ is a type II_1-factor if Γ is an ICC group (that is, all nontrivial conjugacy classes of Γ are infinite). In particular, the free group $\mathbb{F}_n$ on n generators ($2 \leq n \leq \infty$) is an ICC group and thus one obtains the free group factors $\mathcal{L}(\mathbb{F}_n)$ ($2 \leq n \leq \infty$). The crossed product construction is yet another way to obtain von Neumann algebras from a von Neumann algebra $\mathcal{Q}$ and a discrete group Γ acting by $*$-automorphisms on $\mathcal{Q}$ (that is, there exists a group homomorphism $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{Q})$). Briefly, the crossed product algebra $\mathcal{Q} \times_\alpha \Gamma$ is a kind of maximal von Neumann algebra generated by (a copy of) $\mathcal{Q}$ and (a copy of) Γ, subject to the commutation relations $\gamma x \gamma^{-1} = \alpha_\gamma(x)\ \forall \gamma \in \Gamma$ $\forall x \in \mathcal{Q}$. The action $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{Q})$ is called properly outer if each automorphism α_γ, $\gamma \in \Gamma\backslash\{e\}$, is properly outer. An automorphism $\beta \in \mathrm{Aut}(\mathcal{Q})$ is properly outer if for any $x \in \mathcal{Q}$, $xy = \beta(y)x\ \forall y \in \mathcal{Q}$ implies $x = 0$. The action α is said to be ergodic if the fixed-point subalgebra $\mathcal{Q}^\alpha = \{x \in \mathcal{Q} : \alpha_\gamma(x) = x\,\forall \gamma \in \Gamma\}$ is trivial. It is well-known that the crossed product $\mathcal{Q} \times_\alpha \Gamma$ is a factor if the action α is properly outer and if its restriction to the centre of $\mathcal{Q}$ is ergodic. If $\mathcal{Q} \subset \mathcal{P}$ is an inclusion of von Neumann algebras, then the normalizer $N_{\mathcal{P}}(\mathcal{Q})$ of $\mathcal{Q}$ in $\mathcal{P}$ is defined by $N_{\mathcal{P}}(\mathcal{Q}) = \{u \in \mathcal{P} : uu^* = u^*u = 1, u\mathcal{Q}u^* = \mathcal{Q}\}$. The

algebra $\mathcal{Q}$ is said to be regular in $\mathcal{P}$ if $N_{\mathcal{P}}(\mathcal{Q})'' = \mathcal{P}$. For example, $\mathcal{Q}$ is always regular in $\mathcal{Q} \times_\alpha \Gamma$.

2.1. **Correspondences.** The notion of correspondence between two von Neumann algebras (with separable preduals) $\mathcal{P}$ and $\mathcal{A}$ was introduced by A. Connes ([Co2], [Co3], [Po2]). Thus, a correspondence between $\mathcal{P}$ and $\mathcal{A}$ is a pair of mutually commuting normal unital $*$-representations of $\mathcal{P}$ and $\mathcal{A}^o$ (the opposite algebra of $\mathcal{A}$) on the same (separable) infinite dimensional Hilbert space H. Two correspondences ${}_{\mathcal{P}}H_{\mathcal{A}}$ and ${}_{\mathcal{P}}H'_{\mathcal{A}}$ are equivalent if there exists a $(\mathcal{P}, \mathcal{A})$-bilinear isometry from H onto H'. We denote by $\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}}$ the class of ${}_{\mathcal{P}}H_{\mathcal{A}}$ under this equivalence relation and by $\mathrm{Corr}(\mathcal{P}, \mathcal{A})$ the set of all classes of correspondences between $\mathcal{P}$ and $\mathcal{A}$. Let $\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}} \in \mathrm{Corr}(\mathcal{P}, \mathcal{A})$, $\epsilon > 0$, and $F \subset \mathcal{P}$, $E \subset \mathcal{A}$, $S = \{\xi_1, \ldots, \xi_v\} \subset H$ be finite subsets. Define

$$(1) \qquad U\left(\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon, F, E, S\right) = \Big\{\widehat{{}_{\mathcal{P}}K_{\mathcal{A}}} \in \mathrm{Corr}(\mathcal{P}, \mathcal{A}) \,:\, \exists \eta_1, \ldots, \eta_v \in K$$
$$|(b\xi_k a, \xi_l) - (b\eta_k a, \eta_l)| < \epsilon \,\forall b \in F \,\forall a \in E \,\forall 1 \le k, l \le v\Big\}.$$

Define also $V\left(\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon, F, E, S\right)$ as the set of all classes of correspondences $\widehat{{}_{\mathcal{P}}K_{\mathcal{A}}} \in \mathrm{Corr}(\mathcal{P}, \mathcal{A})$ with the property that there exists a surjective isometry $u : H \to K$ such that $||bu(\xi_l)a - u(b\xi_l a)|| < \epsilon$ for all $b \in F$, $a \in E$, $1 \le l \le v$. Then $\mathrm{Corr}(\mathcal{P}, \mathcal{A})$ becomes a topological space with the topology for which the sets $U\left(\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon, F, E, S\right)$ (or equivalently, the sets $V\left(\widehat{{}_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon, F, E, S\right)$) form a basis of neighborhoods. One has also a notion of weak subequivalence: if ${}_{\mathcal{P}}L_{\mathcal{A}}$ and ${}_{\mathcal{P}}L'_{\mathcal{A}}$ are two correspondences between $\mathcal{P}$ and $\mathcal{A}$, we say that ${}_{\mathcal{P}}L_{\mathcal{A}}$ is weakly contained in (or weakly subequivalent to) ${}_{\mathcal{P}}L'_{\mathcal{A}}$ if $\widehat{{}_{\mathcal{P}}L'_{\mathcal{A}}} \in \left\{\widehat{{}_{\mathcal{P}}L_{\mathcal{A}}}\right\}^-$ (we denoted by V^- the closure of $V \subset \mathrm{Corr}(\mathcal{P}, \mathcal{A})$).

We define next a refinement of the above equivalence relation, restricted to subcorrespondences ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M}, \tau)_{\mathcal{A}}$, where $\mathcal{P} \vee \mathcal{A} \subset \mathcal{M}$. Thus, we say that two subcorrespondences ${}_{\mathcal{P}}H_{\mathcal{A}}$, ${}_{\mathcal{P}}H'_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M}, \tau)_{\mathcal{A}}$ are $\mathcal{M}$-equivalent if there exists a unitary $v \in (\mathcal{P} \vee \mathcal{A})' \cap \mathcal{M}$ such that $H = vH'v^*$. We denote by $\widetilde{{}_{\mathcal{P}}H_{\mathcal{A}}}$ the class of ${}_{\mathcal{P}}H_{\mathcal{A}}$ under this equivalence relation and by $\mathrm{Corr}_{\mathcal{M}}(\mathcal{P}, \mathcal{A})$ the set of all classes $\widetilde{{}_{\mathcal{P}}H_{\mathcal{A}}}$ of subcorrespondences ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M}, \tau)_{\mathcal{A}}$. Denote further by $V_{\mathcal{M}}\left(\widetilde{{}_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon, F, E, S\right)$ the set of all classes $\widetilde{{}_{\mathcal{P}}K_{\mathcal{A}}}$ of subcorrespondences ${}_{\mathcal{P}}K_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M}, \tau)_{\mathcal{A}}$ with the property that there exists a unitary $w \in \mathcal{M}$ such that $K = wHw^*$ and $||bw\xi_l w^* a - wb\xi_l aw^*|| < \epsilon$ for all $b \in F$, $a \in E$, $1 \le l \le v$.

Then $\mathrm{Corr}_{\mathcal{M}}(\mathcal{P},\mathcal{A})$ becomes a topological space, endowed with the topology for which the sets $V_{\mathcal{M}}\left(\widetilde{{}_{\mathcal{P}}H_{\mathcal{A}}},\epsilon,F,E,S\right)$ form a basis of neighborhoods. By analogy with the definition of weak subequivalence, one defines the $\mathcal{M}$-weak subequivalence: if ${}_{\mathcal{P}}L_{\mathcal{A}}$ and ${}_{\mathcal{P}}L'_{\mathcal{A}}$ are two subcorrespondences of ${}_{\mathcal{P}}L^2(\mathcal{M},\tau)_{\mathcal{A}}$, we say that ${}_{\mathcal{P}}L_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in (or $\mathcal{M}$-weakly subequivalent to) ${}_{\mathcal{P}}L'_{\mathcal{A}}$ if $\widetilde{{}_{\mathcal{P}}L'_{\mathcal{A}}}\in\left\{\widetilde{{}_{\mathcal{P}}L_{\mathcal{A}}}\right\}^-$ (where V^- denotes here the closure of $V\subset\mathrm{Corr}_{\mathcal{M}}(\mathcal{P},\mathcal{A})$).

2.2. **Free entropy.** We recall but a few results from D. Voiculescu's free probability theory ([Vo1], [Vo2], [Vo3]). Let $\mathcal{P}$ be a finite von Neumann algebra endowed with a normal faithful tracial state $\tau:\mathcal{P}\to\mathbb{C}$. An element $s\in\mathcal{P}$ is called semicircular if $s=s^*$ and if it is distributed according to Wigner's semicircle law:

$$\tau(s^k)=\frac{2}{\pi}\int_{-1}^{1}t^k\sqrt{1-t^2}dt\ \forall k\in\mathbb{N}. \tag{2}$$

A family of $(\mathcal{P}_i)_{i\in I}$ of unital $*$-subalgebras of $\mathcal{P}$ is called free if $\tau(x_1\dots x_m)=0$ whenever $x_k\in\mathcal{P}_{i_k}$, $\tau(x_k)=0$, $\forall 1\le k\le m$, $i_1,\dots,i_m\in I$, $i_1\ne i_2\ne\dots\ne i_m$, $m\in\mathbb{N}$. A family $(X_i)_{i\in I}$ of subsets $X_i\subset\mathcal{P}$ is called free if the family $(*\text{-alg}(\{1\}\cup X_i))_{i\in I}$ is free. The family $(s_i)_{i\in I}$ of elements $s_i\in\mathcal{P}$ is a semicircular system provided that $(\{s_i\})_{i\in I}$ is a free family and if s_i is a semicircular element $\forall i\in I$. If $c\ge 1$ is an integer, we denote by $\mathcal{M}_c(\mathbb{C})$ and $\mathcal{M}_c^{sa}(\mathbb{C})$ the set of all $c\times c$ complex matrices and of all $c\times c$ complex self-adjoint matrices, respectively. We further denote by $\mathcal{U}(c)$ the group of unitaries from $\mathcal{M}_c(\mathbb{C})$, by τ_c the unique unital trace on $\mathcal{M}_c(\mathbb{C})$, and by $||\cdot||_e=\sqrt{c}||\cdot||_2$ the euclidian norm on $\mathcal{M}_c(\mathbb{C})$. The free entropy of $x_1,\dots,x_m\in\mathcal{P}^{sa}$ in the presence of $x_{m+1},\dots,x_{m+n}\in\mathcal{P}^{sa}$ is defined in terms of sets of matricial microstates $\Gamma_R((x_i)_{1\le i\le m}:(x_{m+j})_{1\le j\le n};a,c,\epsilon)\subset(\mathcal{M}_c^{sa}(\mathbb{C}))^m$. Thus, for $a,c\ge 1$ integers and $R,\epsilon>0$, one has the following sequence of definitions:

$$\begin{aligned}&\Gamma_R\left((x_i)_{1\le i\le m}:(x_{m+j})_{1\le j\le n};a,c,\epsilon\right)\\&\quad=\{(A_i)_{1\le i\le m}\in(\mathcal{M}_c^{sa}(\mathbb{C}))^m\ :\ \exists\,(A_{m+j})_{1\le j\le n}\in(\mathcal{M}_c^{sa}(\mathbb{C}))^n\text{ s.t.}\\&\quad|\tau\left(x_{i_1}\dots x_{i_l}\right)-\tau_c(A_{i_1}\dots A_{i_l})|<\epsilon\,,\ ||A_k||\le R\\&\quad\forall\,1\le i_1,\dots,i_l\le m+n\,\forall 1\le l\le a\,\forall 1\le k\le m+n\}\,,\end{aligned} \tag{3}$$

$$\begin{aligned}&\chi_R((x_i)_{1\le i\le m}:(x_{m+j})_{1\le j\le n};a,c,\epsilon)\\&\qquad=\log\mathrm{vol}_{mc^2}(\Gamma_R((x_i)_{1\le i\le m}:(x_{m+j})_{1\le j\le n};a,c,\epsilon)),\end{aligned} \tag{4}$$

$$\chi_R\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}; a,\epsilon\right) = \limsup_{c\to\infty}\left(\frac{1}{c^2}\chi_R\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}; a,c,\epsilon\right) + \frac{m}{2}\log c\right), \tag{5}$$

$$\chi_R\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}\right) = \inf_{a,\epsilon}\chi_R\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}; a,\epsilon\right), \tag{6}$$

$$\chi\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}\right) = \sup_R \chi_R\left((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n}\right) \tag{7}$$

($\mathrm{vol}_{mc^2}(\cdot)$ denotes the Lebesgue measure on $(\mathcal{M}_c^{sa}(\mathbb{C}))^m \simeq \mathbb{R}^{mc^2}$). The last quantity $\chi((x_i)_{1\le i\le m} : (x_{m+j})_{1\le j\le n})$ is called the free entropy of $(x_i)_{1\le i\le m}$ in the presence of $(x_{m+j})_{1\le j\le n}$. If $n=0$, then it is simply called the free entropy of $(x_i)_{1\le i\le m}$, denoted $\chi(x_1,\ldots,x_m)$. The free entropy of $(x_i)_{1\le i\le m}$ in the presence of $(x_{m+j})_{1\le j\le n}$ is equal to the free entropy of $(x_i)_{1\le i\le m}$ if $\{x_{m+1},\ldots,x_{m+n}\}\subset\{x_1,\ldots,x_m\}''$. For a single self-adjoint element $x\in\mathcal{P}$ with distribution μ one has

$$\chi(x) = \frac{3}{4}+\frac{1}{2}\log 2\pi + \int\int \log|s-t|d\mu(s)d\mu(t). \tag{8}$$

Also, if $(x_i)_{1\le i\le m}$ is a free family, then $\chi(x_1,\ldots,x_m)=\chi(x_1)+\ldots+\chi(x_m)$. In particular, a finite semicircular system has finite free entropy. The modified free entropy dimension of $(x_i)_{1\le i\le m}$ is defined as follows:

$$\delta_0((x_i)_{1\le i\le m}) = m + \limsup_{\omega\to 0}\frac{\chi((x_i+\omega s_i)_{1\le i\le m} : (s_i)_{1\le i\le m})}{|\log\omega|}, \tag{9}$$

while its free entropy dimension is

$$\delta((x_i)_{1\le i\le m}) = m + \limsup_{\omega\to 0}\frac{\chi((x_i+\omega s_i)_{1\le i\le m})}{|\log\omega|}, \tag{10}$$

where $(x_i)_{1\le i\le m}$ and the semicircular system $(s_i)_{1\le i\le m}$ are free. Both free entropy dimensions can be determined with the following formulae if the family $(x_i)_{1\le i\le m}$ is free:

$$\delta_0((x_i)_{1\le i\le m}) = \delta((x_i)_{1\le i\le m}) = \sum_{i=1}^m \delta(x_i), \tag{11}$$

$$\delta(x) = 1 - \sum_{s\in\mathbb{R}}(\mu(\{s\}))^2. \tag{12}$$

We mention in this context the important Semicontinuity Problem ([Vo3]): if x_i is the SOT-limit of $x_i^{(p)}\in\mathcal{P}$ as $p\to\infty$ (for all $1\le i\le m$),

does it follow then that $\liminf_{p\to\infty}\delta_0((x_i^{(p)})_{1\le i\le m})\ge\delta_0((x_i)_{1\le i\le m})$? An affirmative answer to this question would imply ([Vo3]) the nonisomorphism of $\mathcal{L}(\mathbb{F}_n)$ and $\mathcal{L}(\mathbb{F}_m)$ for $n\neq m$.

3. Estimate of free entropy

Lemma 3.1 gives an estimate for the free entropy of an arbitrary system of generators of $\mathcal{M}$ which can be ω-approximated in the 2-norm by certain noncommutative polynomials. Typically, this situation is encountered under the hypothesis of Theorem 4.1, where the ω-approximations hold for all $\omega>0$.

Lemma 3.1. *Let $x_1,\dots,x_m$ be self-adjoint generators of a II_1-factor $(\mathcal{M},\tau)$. Assume that $\mathcal{N}$ is a subfactor of $\mathcal{M}$ and $\mathcal{P}=\mathcal{P}_1\vee\mathcal{P}_2\subset\mathcal{N}$ is a von Neumann subalgebra, where $\mathcal{P}_1=\mathcal{R}_1'\cap\mathcal{P}$, $\mathcal{P}_2=\mathcal{R}_2'\cap\mathcal{P}$ and $\mathcal{R}_1$, $\mathcal{R}_2$ are mutually commuting hyperfinite subfactors of $\mathcal{P}$. Assume moreover that there exist self-adjoint elements $m_j^{(e)}, z_k\in\mathcal{M}$ (for $1\le j\le r+1$, $1\le e\le 2$, $1\le k\le 2v$), mutually orthogonal projections $p_q\in\mathcal{M}$ (for $1\le q\le u$), projections $(p^{(t)})_t\subset\mathcal{P}_1$, $(q^{(s)})_s\subset\mathcal{P}_2$ of trace $\frac{1}{2}$, and noncommutative polynomials $\Phi_{ji}^{(e)}((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q)$ which are linear combinations of monomials of the form $p^{(t_1)}q^{(s_1)}\dots p^{(t_a)}q^{(s_a)}z_kp_q$, such that for some $\omega>0$ and for all $1\le i\le m$*

$$
\begin{aligned}
(13)\qquad &\Bigg|\Bigg|x_i-\frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\Big(m_j^{(e)}\Phi_{ji}^{(e)}\left((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q\right)\\
&\qquad+\Phi_{ji}^{(e)}\left((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q\right)^*m_j^{(e)}\Big)\Bigg|\Bigg|_2<\omega\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
(14)\qquad \chi\left((x_i)_{1\le i\le m}\right)&=\chi\Bigg((x_i)_{1\le i\le m}:\left(m_j^{(e)}\right)_{j,e},(p^{(t)})_t,(q^{(s)})_s,\\
&\qquad(z_k)_k,(p_q)_q\Bigg)\le C(m,r,v,K)+(m-2r-2v-4)\log\omega\,.
\end{aligned}
$$

where $C(m,r,v,K)$ is a constant depending only on m, r, v, and $K=1+\max_{i,j,e}\left\{\left|\left|\Phi_{ji}^{(e)}\left((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q\right)\right|\right|_2,||x_i||,\left|\left|m_j^{(e)}\right|\right|\right\}$.

Proof. Let $c_0\ge 1$ be a fixed integer. Let $\mathcal{M}_1\subset\mathcal{R}_1$ and $\mathcal{M}_2\subset\mathcal{R}_2$ be two subalgebras isomorphic to $\mathcal{M}_{c_0}(\mathbb{C})$ and let $(e_{gh})_{g,h}$, $(f_{gh})_{g,h}$ be matrix units for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Consider a matricial microstate

$$
\left((A_i)_i,(M_j^{(e)})_{j,e},(P^{(t)})_t,(Q^{(s)})_s,(Z_k)_k,(P_q)_q,(E_{gh})_{g,h},(F_{gh})_{g,h}\right)
$$

from the set of matricial microstates

$$\Gamma_R\left((x_i)_i,(m_j^{(e)})_{j,e},(p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q,(e_{gh})_{g,h},(f_{gh})_{g,h};a,c,\epsilon\right).$$

We can assume ([Vo3]) that $||A_i||,||M_j^{(e)}||\leq K$. If a is large and $\epsilon>0$ is small enough, then $\forall 1\leq i\leq m$

$$\left|\left|A_i-\frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\left(M_j^{(e)}\Phi_{ji}^{(e)}\left((P^{(t)})_t,(Q^{(s)})_s,(Z_k)_k,(P_q)_q\right)\right.\right.\right.$$
$$\left.\left.\left.+\Phi_{ji}^{(e)}\left((P^{(t)})_t,(Q^{(s)})_s,(Z_k)_k,(P_q)_q\right)^*M_j^{(e)}\right)\right|\right|_2<\omega \tag{15}$$

and $\left|\left|\Phi_{ji}^{(e)}((P^{(t)})_t,(Q^{(s)})_s,(Z_k)_k,(P_q)_q)\right|\right|_2\leq K$ for all i,j,e. For any $\delta>0$ there exists an injective $*$-homomorphism $\alpha_c:\mathcal{M}_1\vee\mathcal{M}_2\to\mathcal{M}_c(\mathbb{C})$ such that

$$||\alpha_c(e_{gh})-E_{gh}||_2<\delta,\ ||\alpha_c(e_{gh})-E_{gh}||_2<\delta\,\forall g,h$$

for large $a\in\mathbb{N}$ and small $\epsilon>0$, but independently of c. The conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_1'\cap\mathcal{M}$ is given by

$$E_{\mathcal{M}_1'\cap\mathcal{M}}(x)=\frac{1}{c_0}\sum_{g,h=1}^{c_0}e_{gh}xe_{hg}=\Xi(x,(e_{gh})_{g,h}).$$

Denote $P_0^{(t)}=\Xi(P^{(t)},(\alpha_c(e_{gh}))_{g,h})\in\alpha_c(\mathcal{M}_1)'\cap\mathcal{M}_c(\mathbb{C})$. For any $\delta_1>0$ and any $a_1\in\mathbb{N}$, since $p^{(t)}=E_{\mathcal{M}_1'\cap\mathcal{M}}(p^{(t)})=\Xi(p^{(t)},(e_{gh})_{g,h})$, it follows that

$$\left|\tau_c\left((P_0^{(t)})^l\right)-\tau\left((p^{(t)})^l\right)\right|<\delta_1\forall 1\leq l\leq a_1$$

if $\epsilon,\delta>0$ are small and $a\in\mathbb{N}$ is large enough. Given $\delta_2>0$, if δ_1 is sufficiently small and a_1 is sufficiently large, there exists ([Vo2]) a projection $P_1^{(t)}\in\alpha_c(\mathcal{M}_1)'\cap\mathcal{M}_c(\mathbb{C})$ of rank $\frac{c}{2}$, such that $\left|\left|P_1^{(t)}-P_0^{(t)}\right|\right|_2<\delta_2$. Note that

$$\left|\left|P_0^{(t)}-P^{(t)}\right|\right|_2^2=\tau_c\left(\left(P^{(t)}-\Xi(P^{(t)},(\alpha_c(e_{gh}))_{g,h})\right)^2\right)$$

and also

$$\tau\left(\left(p^{(t)}-\Xi(p^{(t)},(e_{gh})_{g,h})\right)^2\right)=0,$$

hence $\left|\left|P^{(t)}-P_0^{(t)}\right|\right|_2<\delta_2$ and thus $\left|\left|P^{(t)}-P_1^{(t)}\right|\right|_2<2\delta_2$ if ϵ,δ are small enough and a is sufficiently large. In this way we can find projections $(P_1^{(t)})_t\subset\alpha_c(\mathcal{M}_1)'\cap\mathcal{M}_c(\mathbb{C})$, $(Q_1^{(s)})_s\subset\alpha_c(\mathcal{M}_2)'\cap\mathcal{M}_c(\mathbb{C})$, of

rank $\frac{c}{2}$, such that $\left|\left|P^{(t)} - P_1^{(t)}\right|\right|_2 < 2\delta_2$ and $\left|\left|Q^{(s)} - Q_1^{(s)}\right|\right|_2 < 2\delta_2$ for all t, s. If δ_2 is small enough then we have moreover $\forall 1 \leq i \leq m$

$$\left|\left| A_i - \frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\left(M_j^{(e)}\Phi_{ji}^{(e)}\left((P_1^{(t)})_t, (Q_1^{(s)})_s, (Z_k)_k, (P_q)_q\right)\right.\right.\right.$$
$$\left.\left.\left.+\Phi_{ji}^{(e)}\left((P_1^{(t)})_t, (Q_1^{(s)})_s, (Z_k)_k, (P_q)_q\right)^* M_j^{(e)}\right)\right|\right|_2 < \omega. \tag{16}$$

Fix two copies $\mathcal{G}_1(c) \subset \alpha_c(\mathcal{M}_1)' \cap \mathcal{M}_c(\mathbb{C})$ and $\mathcal{G}_2(c) \subset \alpha_c(\mathcal{M}_2)' \cap \mathcal{M}_c(\mathbb{C})$ of the Grassmann manifold $\mathcal{G}\left(\frac{c}{c_0}, \frac{c}{2c_0}\right)$ and note that there exists a unitary $U \in \mathcal{U}(c)$ such that $UP_1^{(t)}U^* \in \mathcal{G}_1(c)$ and $UQ_1^{(s)}U^* \in \mathcal{G}_2(c)$ for all t, s. Lemma 4.3 in [Vo2] implies that given $\delta_3 > 0$, there exist $a', c' \in \mathbb{N}$, $\epsilon_1 > 0$ such that if $c \geq c'$ and if $(P_1, \dots, P_u) \in \Gamma_R((p_q)_q; a', c, \epsilon_1)$, then there exist mutually orthogonal projections $P_1', \dots, P_u' \subset \mathcal{M}_c^{sa}(\mathbb{C})$ such that $\text{rank}(P_q') = [\tau(p_q)c]$ and $\left|\left|P_q - P_q'\right|\right|_2 < \delta_3$ $\forall 1 \leq q \leq u$. Let $(S_q)_q$ be fixed mutually orthogonal projections with $\text{rank}(S_q) = [\tau(p_q)c]$ $\forall 1 \leq q \leq u$ and let $W \in \mathcal{U}(c)$ be a unitary such that $P_q' = W^* S_q W$ $\forall 1 \leq q \leq u$. If $\delta_3 > 0$ is sufficiently small, then one has

$$\left|\left| UA_iW^* - \frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\left(UM_j^{(e)}U^*\Phi_{ji}^{(e)}\left((UP_1^{(t)}U^*)_t, (UQ_1^{(s)}U^*)_s,\right.\right.\right.\right.$$
$$(UZ_kW^*)_k, (S_q)_q\big) + UW^*\Phi_{ji}^{(e)}\big((UP_1^{(t)}U^*)_t, (UQ_1^{(s)}U^*)_s,$$
$$\left.\left.\left.\left.(UZ_kW^*)_k, (S_q)_q\right)^* UM_j^{(e)}U^*(UW^*)\right)\right|\right|_2 < \omega \,\forall 1 \leq i \leq m. \tag{17}$$

Consider a minimal θ-net $(V_b)_{b \in B(c,K)}$ in $\{B \in \mathcal{M}_c^{sa}(\mathbb{C}) : ||B|| \leq K\}$ and a minimal $\frac{\omega}{2K}$-net $(U_t)_{t \in T(c)}$ in $\mathcal{U}(c)$ with respect to the uniform norm. Let also $(G_a^{(1)})_{a \in A(c)}$ and $(G_a^{(2)})_{a \in A(c)}$ be two minimal η-nets (relative to the euclidian norm induced from $\mathcal{M}_c(\mathbb{C})$) in $\mathcal{G}_1(c)$ and respectively, $\mathcal{G}_2(c)$. From [Sz] we have $|T(c)| \leq (\frac{2CK}{\omega})^{c^2}$, $|B(c, K)| \leq (\frac{CK}{\theta})^{c^2+c}$, $|A(c)| \leq (\frac{C\sqrt{c}}{\eta})^{\frac{c^2}{2c_0^2}}$, where C is a universal constant. There exist indices

$t, s \in T(c)$, $b(j,e) \in B(c,K)$, $a(t), a(s) \in A(c)$ such that $\forall 1 \le i \le m$

$$\begin{aligned}(18)\quad & \Bigg\| U_t A_i W_s^* - \frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\Bigg(V_{b(j,e)}\Phi_{ji}^{(e)}\Bigg(\left(G_{a(t)}^{(1)}\right)_t, \left(G_{a(s)}^{(2)}\right)_s, (T_k)_k, \\ & (S_q)_q\Bigg) + U_t W_s^* \Phi_{ji}^{(e)}\Bigg(\left(G_{a(t)}^{(1)}\right)_t, \left(G_{a(s)}^{(2)}\right)_s, (T_k)_k, (S_q)_q\Bigg)^* \\ & \cdot V_{b(j,e)} U_t W_s^*\Bigg)\Bigg\|_e < \omega\sqrt{c} + \omega\sqrt{c} + \frac{1}{2}\cdot 2(r+1)\cdot\Big[\theta K\sqrt{c} \\ & + D(\Phi)K\eta\sqrt{\alpha+\beta} + 2\cdot\frac{\omega}{2K}K^2\sqrt{c} + D(\Phi)K\eta\sqrt{\alpha+\beta} \\ & + K\left(\theta\sqrt{c} + 2K\frac{\omega}{2K}\sqrt{c}\right)\Big] = 2\left[K(r+1)+1\right]\omega\sqrt{c} \\ & + 2\theta(r+1)K\sqrt{c} + 2D(\Phi)K\eta(r+1)\sqrt{\alpha+\beta}\,,\end{aligned}$$

where $D(\Phi)$ is a Lipschitz constant depending on the Φ's, and α, β are the number of $P_1^{(t)}$'s and $Q_1^{(s)}$'s. Choose $\theta = \frac{\omega}{2K(r+1)}$ and $\eta = \frac{\omega\sqrt{c}}{2D(\Phi)K(r+1)\sqrt{\alpha+\beta}}$, so that $|B(c,K)| \le \left(\frac{2CK^2(r+1)}{\omega}\right)^{c^2+c}$ and $|A(c)| \le \left(\frac{2CD(\Phi)K(r+1)\sqrt{\alpha+\beta}}{\omega}\right)^{\frac{c^2}{2c_0^2}}$. The volume of the set of matricial microstates can be estimated as follows:

$$\begin{aligned}(19)\quad & \mathrm{vol}_{mc^2}\Big(\Gamma_R\Big((x_i)_i : \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q, (e_{gh})_{g,h}, \\ & (f_{gh})_{g,h}; a, c, \epsilon\Big)\Big) \le \left(\frac{2CK}{\omega}\right)^{2c^2}\cdot\left(\frac{2CK^2(r+1)}{\omega}\right)^{2(r+1)(c^2+c)} \\ & \cdot\left(\frac{2CD(\Phi)K(r+1)\sqrt{\alpha+\beta}}{\omega}\right)^{\frac{c^2(\alpha+\beta)}{2c_0^2}}\cdot \mathrm{vol}_{d_c}\left(0, (K+\mu)\sqrt{mc}\right) \\ & \cdot \mathrm{vol}_{mc^2-d_c}\left(0, \mu\sqrt{mc}\right),\end{aligned}$$

where $\mu = 2\omega\left[K(r+1)+2\right]$ and d_c denotes the dimension of the range of the linear map that sends $(T_k)_k$ to

$$\begin{aligned}(20)\quad & \Bigg(\frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\Bigg(U_t^* V_{b(j,e)}\Phi_{ji}^{(e)}\left(\left(G_{a(t)}^{(1)}\right)_t, \left(G_{a(s)}^{(2)}\right)_s, (T_k)_k, (S_q)_q\right) W_s \\ & + W_s^*\Phi_{ji}^{(e)}\left(\left(G_{a(t)}^{(1)}\right)_t, \left(G_{a(s)}^{(2)}\right)_s, (T_k)_k, (S_q)_q\right)^* V_{b(j,e)} U_t\Bigg)\Bigg)_{1\le i\le m}.\end{aligned}$$

Since $(x_i)_{1\le i\le m}$ generates $\mathcal{M}$, the last inequality implies the free entropy estimate

$$
\begin{aligned}
(21)\quad \chi\left((x_i)_{1\le i\le m}\right) &= \chi\Big((x_i)_{1\le i\le m} : \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, \\
&\qquad (z_k)_k, (p_q)_q\Big) \\
&= \chi\Big((x_i)_{1\le i\le m} : \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, \\
&\qquad (z_k)_k, (p_q)_q, (e_{gh})_{g,h}, (f_{gh})_{g,h}\Big) \\
&\le C(m,r,v,K) + (m-2r-2v-4)\log\omega\,.
\end{aligned}
$$

□

4. Applications

The main application of the free entropy estimate from the previous section is Theorem 4.1: the (modified) free entropy dimension of any set of generators of $\mathcal{M}$ is $\le 2r+2v+4$ if the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in a finitely generated subcorrespondence ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M},\tau)_{\mathcal{A}}$, where $\mathcal{P}\subset\mathcal{N}$ is generated by the (diffuse) relative commutants of two commuting copies of the hyperfinite II_1-factor $\mathcal{R}$, $\mathcal{A}\subset\mathcal{N}$ is an abelian subalgebra, v is the number of vectors which span ${}_{\mathcal{P}}H_{\mathcal{A}}$, and r is the integer part of the Jones index $[\mathcal{M}:\mathcal{N}]$. The results concerning the free group subfactors (such as the absence of regular non-prime subfactors) are listed in Theorem 4.2. We proceed with two short Lemmas, 4.1 and 4.2, which will be used further in the proofs of Theorems 4.1 and 4.2, respectively.

Lemma 4.1. *Let ${}_{\mathcal{P}}H_{\mathcal{A}}$ and ${}_{\mathcal{P}}K_{\mathcal{A}}$ be subcorrespondences of ${}_{\mathcal{P}}L^2(\mathcal{M},\tau)_{\mathcal{A}}$ such that ${}_{\mathcal{P}}H_{\mathcal{A}}$ is generated by v vectors and ${}_{\mathcal{P}}K_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in ${}_{\mathcal{P}}H_{\mathcal{A}}$. Then $\forall\epsilon>0$ $\forall\lambda_1,\dots,\lambda_m\in K$ $\exists$ unitary $u\in\mathcal{M}$ $\exists\kappa_1,\dots,\kappa_v\in K$, $\exists$ finite $\left\{b_{j,l}^{(i)}\right\}_{i,j,l}\subset\mathcal{P}$, $\exists$ finite $\left\{a_{j,l}^{(i)}\right\}_{i,j,l}\subset\mathcal{A}$ such that $K=uKu^*$ and*

$$\left\|u^*\lambda_i u - \sum_{j,l} b_{j,l}^{(i)}\kappa_l a_{j,l}^{(i)}\right\| < \epsilon\,\forall 1\le i\le m.$$

Proof. Note first that ${}_{\mathcal{P}}K_{\mathcal{A}}$ $\mathcal{M}$-weakly contained in ${}_{\mathcal{P}}H_{\mathcal{A}}$ implies $\widetilde{{}_{\mathcal{P}}K_{\mathcal{A}}}\in V_{\mathcal{M}}\left(\widetilde{{}_{\mathcal{P}}H_{\mathcal{A}}},\epsilon_0,F,E,S\right)$ for all $\epsilon_0>0$ and all finite subsets $F\subset\mathcal{P}$, $E\subset\mathcal{A}$, $S\subset H$. This shows in particular that there exists a unitary $w\in\mathcal{M}$ such that $K=wHw^*$, hence $\lambda_1=w\eta_1w^*,\dots,\lambda_v=w\eta_vw^*$ for some $\eta_1,\dots,\eta_v\in H$. Since ${}_{\mathcal{P}}H_{\mathcal{A}}$ is generated by v vectors, there exist

$\xi_1, \ldots, \xi_v \in H$ such that

$$_{\mathcal{P}}H_{\mathcal{A}} =_{\mathcal{P}} \overline{\mathrm{sp}}^{||\cdot||} \left(\mathcal{P}\xi_1\mathcal{A} + \ldots + \mathcal{P}\xi_v\mathcal{A}\right)_{\mathcal{A}}.$$

Therefore, given $\epsilon_1 > 0$, there exist finite subsets $F = \left\{b_{j,l}^{(i)}\right\}_{i,j,l} \subset \mathcal{P}$, $E = \left\{a_{j,l}^{(i)}\right\}_{i,j,l} \subset \mathcal{A}$ such that

$$\left|\left|\eta_i - \sum_{j,l} b_{j,l}^{(i)}\xi_l a_{j,l}^{(i)}\right|\right| < \epsilon_1 \,\forall 1 \leq i \leq m.$$

Given $\epsilon_2 > 0$, since $\widetilde{_{\mathcal{P}}K_{\mathcal{A}}} \in V_{\mathcal{M}}\left(\widetilde{_{\mathcal{P}}H_{\mathcal{A}}}, \epsilon_2, F, E, \{\xi_l\}_l\right)$, there exists a unitary $v \in \mathcal{M}$ such that $K = vHv^*$ and

$$||bv\xi_l v^* a - vb\xi_l av^*|| < \epsilon_2 \,\forall b \in F \,\forall a \in E \,\forall 1 \leq l \leq v.$$

Let $\kappa_l = v\xi_l v^*$ $\forall 1 \leq l \leq v$ and note that one has ($\forall 1 \leq i \leq m$) the following estimate:

$$\begin{aligned}
(22)\quad \left|\left|vw^*\lambda_i wv^* - \sum_{j,l} b_{j,l}^{(i)}\kappa_l a_{j,l}^{(i)}\right|\right| &= \left|\left|\eta_i - \sum_{j,l} v^* b_{j,l}^{(i)} v\xi_l v^* a_{j,l}^{(i)} v\right|\right| \\
&\leq \left|\left|\eta_i - \sum_{j,l} b_{j,l}^{(i)}\xi_l a_{j,l}^{(i)}\right|\right| + \sum_{j,l}\left|\left|b_{j,l}^{(i)}\xi_l a_{j,l}^{(i)} - v^* b_{j,l}^{(i)} v\xi_l v^* a_{j,l}^{(i)} v\right|\right| \\
&< \epsilon_1 + \sum_{j,l}\left|\left|vb_{j,l}^{(i)}\xi_l a_{j,l}^{(i)} v^* - b_{j,l}^{(i)} v\xi_l v^* a_{j,l}^{(i)}\right|\right|.
\end{aligned}$$

The last term in (22) is smaller than ϵ if ϵ_1 and ϵ_2 are sufficiently small. □

Lemma 4.2. *Let $\mathcal{P}$ be a von Neumann algebra with a matrix unit $(e_{ij})_{1\leq i,j\leq k} \subset \mathcal{P}$ and let also $\mathcal{A}$ be an abelian algebra with a projection $q \in \mathcal{A}$. Assume that the correspondence $_{\mathcal{P}}H_{\mathcal{A}}$ is finitely generated: $H = \overline{sp}^{||\cdot||}(\mathcal{P}\xi_1\mathcal{A} + \ldots + \mathcal{P}\xi_v\mathcal{A})$ for some $\xi_1, \ldots, \xi_v \in H$. Then the correspondence $_{\mathcal{P}_p}(pHq)_{\mathcal{A}_q}$ is also finitely generated:*

$$pHq = \overline{sp}^{||\cdot||}\left(\sum_{i=1}^{k}\sum_{l=1}^{v} \mathcal{P}_p\xi_{il}\mathcal{A}_q\right),$$

where $p = e_{11}$ and $\xi_{il} = e_{1i}\xi_l q$ $\forall 1 \leq i \leq k$ $\forall 1 \leq l \leq v$.

Proof.

$$(23)\ pHq = \overline{\mathrm{sp}}^{||\cdot||}\left(\sum_{l=1}^{v} p\mathcal{P}\xi_l\mathcal{A}q\right) = \overline{\mathrm{sp}}^{||\cdot||}\left(\sum_{i=1}^{k}\sum_{l=1}^{v} p\mathcal{P}e_{ii}\xi_l\mathcal{A}q\right)$$
$$= \overline{\mathrm{sp}}^{||\cdot||}\left(\sum_{i=1}^{k}\sum_{l=1}^{v} p\mathcal{P}e_{i1}e_{1i}\xi_l q\mathcal{A}\right) = \overline{\mathrm{sp}}^{||\cdot||}\left(\sum_{i=1}^{k}\sum_{l=1}^{v} \mathcal{P}_p\xi_{il}\mathcal{A}_q\right).$$

□

Theorem 4.1. *Let $(\mathcal{M},\tau)$ be a II_1-factor generated by the self-adjoint elements $x_1,\dots,x_m$. Assume that $\mathcal{N}\subset\mathcal{M}$ is a subfactor with the integer part of the Jones index $[\mathcal{M}:\mathcal{N}]$ equal to r, $\mathcal{A}\subset\mathcal{N}$ is an abelian subalgebra, and $\mathcal{P}\subset\mathcal{N}$ is a subalgebra such that $\mathcal{P}=\mathcal{P}_1\vee\mathcal{P}_2$, where $\mathcal{P}_1=\mathcal{R}_1'\cap\mathcal{P}$, $\mathcal{P}_2=\mathcal{R}_2'\cap\mathcal{P}$ and $\mathcal{R}_1,\mathcal{R}_2\subset\mathcal{P}$ are mutually commuting hyperfinite subfactors. Assume moreover that $\mathcal{P}_1$, $\mathcal{P}_2$ are diffuse von Neumann subalgebras and that the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is $\mathcal{M}$-weakly contained in a subcorrespondence ${}_{\mathcal{P}}H_{\mathcal{A}}$ of ${}_{\mathcal{P}}L^2(\mathcal{M},\tau)_{\mathcal{A}}$, generated by v vectors. Then*

$$(24)\qquad \delta_0(x_1,\dots,x_m)\le 2r+2v+4\,.$$

Proof. Note first that one has $\delta_0(x_1,\dots,x_m)\le m$ ([Vo3]) and thus one can assume $m>2r+2v+4$. There exist ([PiPo]) $m_1,\dots,m_{r+1}\in\mathcal{M}$ such that

$$(25)\qquad x=\sum_{j=1}^{r+1} m_jE_{\mathcal{N}}(m_j^*x)\,\forall x\in\mathcal{M},$$

where $E_{\mathcal{N}}:\mathcal{M}\to\mathcal{N}$ is the conditional expectation onto $\mathcal{N}$. Use Lemma 4.1 to conclude that for every $\epsilon>0$ there exist a unitary $u\in\mathcal{M}$, self-adjoint vectors $\eta_1,\dots,\eta_{2v}\in L^2(\mathcal{N},\tau)^{sa}$ and finite subsets $\left\{b_{p,k}^{(i,j)}\right\}_{i,j,p,k}\subset\mathcal{P}$, $\left\{a_{p,k}^{(i,j)}\right\}_{i,j,p,k}\subset\mathcal{A}$ such that

$$\left|\left|u^*E_{\mathcal{N}}(m_j^*x_i)u-\sum_{k=1}^{2v}\sum_{p=1}^{l} b_{p,k}^{(i,j)}\eta_k a_{p,k}^{(i,j)}\right|\right|_2<\epsilon\,\forall 1\le i\le m\,\forall 1\le j\le r+1.$$

Since $u\mathcal{A}u^*$ is abelian, there exist projections $p_1,\dots,p_u\in u\mathcal{A}u^*$ of sum 1 such that every $ua_{p,k}^{(i,j)}u^*$ is approximated sufficiently well in the $||\cdot||$-norm by linear combinations of these projections. Being diffuse, both $u\mathcal{P}_1u^*$ and $u\mathcal{P}_2u^*$ are generated by their projections of trace $\frac{1}{2}$, hence each $ub_{p,k}^{(i,j)}u^*$ is the SOT-limit of a sequence of noncommutative polynomials $\Psi_{p,k}^{(i,j)}\left((p^{(t)})_t,(q^{(s)})_s\right)$ in projections of trace $\frac{1}{2}$, $(p^{(t)})_t\subset$

$u\mathcal{P}_1u^*$, $(q^{(s)})_s \subset u\mathcal{P}_2u^*$. Moreover, $u\mathcal{N}^{sa}u^*$ is dense in $L^2(u\mathcal{N}u^*, \tau)^{sa}$ hence there exist $z_1, \ldots, z_{2v}$ self-adjoint elements of $u\mathcal{N}u^*$ such that

(26)
$$\Lambda_{ji}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right) = \sum_{p=1}^{l}\sum_{k=1}^{2v}\sum_{q=1}^{u} \Psi_{p,k}^{(i,j)}\left((p^{(t)})_t, (q^{(s)})_s\right) z_k p_q$$

is sufficiently close to $E_{\mathcal{N}}(m_j^* x_i)$ in the $||\cdot||_2$-norm. Therefore each x_i can be approximated arbitrarily well in the $||\cdot||_2$-norm by elements of the form

$$\sum_{j=1}^{r+1} m_j \Lambda_{ji}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right). \tag{27}$$

Denote $m_j^{(1)} = \frac{m_j + m_j^*}{2}$, $m_j^{(2)} = \frac{m_j - m_j^*}{2\sqrt{-1}}$, and

$$\begin{aligned}\Phi_{ji}^{(1)}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right) &= \Lambda_{ji}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right) \\ &= -\sqrt{-1}\Phi_{ji}^{(2)}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right).\end{aligned} \tag{28}$$

Hence for every $\omega > 0$ there exist polynomials $\Phi_{ji}^{(e)}\big((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\big)$ that are linear combinations of monomials of the form $p^{(t_1)}q^{(s_1)} \ldots p^{(t_a)}q^{(s_a)}z_k p_q$ such that

$$\begin{aligned}\Bigg|\Bigg| x_i - \frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\Big(m_j^{(e)}\Phi_{ji}^{(e)}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right) \\ + \Phi_{ji}^{(e)}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right)^* m_j^{(e)}\Big)\Bigg|\Bigg|_2 < \omega\, \forall 1 \le i \le m.\end{aligned} \tag{29}$$

Also, one can assume that $\left(\left|\left|\Phi_{ji}^{(e)}\left((p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right)\right|\right|_2\right)_{i,j,e}$ are uniformly bounded by a constant D depending only on $(||m_j^* x_i||)_{i,j}$. Consider a semicircular system $(s_i)_{1\le i\le m}$, free from $(x_i)_{1\le i\le m}$. Note that since $\left(m_j^{(e)}\right)_{j,e}$, $(p^{(t)})_t$, $(q^{(s)})_s$, $(z_k)_k$, $(p_q)_q$ are all contained in $\{x_i + \omega s_i, s_i : 1 \le i \le m\}''$, one has ([Vo3])

$$\begin{aligned}\chi\left((x_i + \omega s_i)_{1\le i\le m} : (s_i)_{1\le i\le m}\right) = \chi\Bigg((x_i + \omega s_i)_{1\le i\le m} : (s_i)_{1\le i\le m}, \\ \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\Bigg) \le \chi\Bigg((x_i + \omega s_i)_{1\le i\le m} : \\ \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\Bigg)\, \forall 1 \le i \le m.\end{aligned} \tag{30}$$

The inequalities (29) imply

$$\left\|x_i + \omega s_i - \frac{1}{2}\sum_{e=1}^{2}\sum_{j=1}^{r+1}\Big(m_j^{(e)}\Phi_{ji}^{(e)}\left((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q\right) + \Phi_{ji}^{(e)}\left((p^{(t)})_t,(q^{(s)})_s,(z_k)_k,(p_q)_q\right)^* m_j^{(e)}\Big)\right\|_2 < 2\omega \; \forall 1 \le i \le m \tag{31}$$

hence, by (30) and the free entropy estimate (14) from Lemma 3.1,

$$\chi\left((x_i+\omega s_i)_{1\le i\le m} : (s_i)_{1\le i\le m}\right) \le C(m,r,v,K) + (m-2r-2v-4)\log 2\omega\,.$$

The estimate for the (modified) free entropy dimension follows now immediately:

$$\begin{aligned}\delta_0(x_1,\ldots,x_m) &= m + \limsup_{\omega\to 0}\frac{\chi\left((x_i+\omega s_i)_{1\le i\le m} : (s_i)_{1\le i\le m}\right)}{|\log\omega|} \\ &\le m + \limsup_{\omega\to 0}\frac{C(m,r,v,K)+(m-2r-2v-4)\log 2\omega}{|\log\omega|} \\ &= 2r+2v+4\,.\end{aligned} \tag{32}$$

□

Corollary 4.1. *Let $(\mathcal{M},\tau)$ be a II_1-factor generated by the self-adjoint elements $x_1,\ldots,x_m$. Assume that $\mathcal{N}\subset\mathcal{M}$ is a subfactor with the integer part of the Jones index $[\mathcal{M}:\mathcal{N}]$ equal to r, $\mathcal{A}\subset\mathcal{N}$ is an abelian subalgebra, and $\mathcal{P}\subset\mathcal{N}$ is a subalgebra such that $\mathcal{P}=\mathcal{P}_1\vee\mathcal{P}_2$, where $\mathcal{P}_1=\mathcal{R}_1'\cap\mathcal{P}$, $\mathcal{P}_2=\mathcal{R}_2'\cap\mathcal{P}$ and $\mathcal{R}_1,\mathcal{R}_2\subset\mathcal{P}$ are mutually commuting hyperfinite subfactors. Assume moreover that $\mathcal{P}_1$, $\mathcal{P}_2$ are diffuse von Neumann subalgebras and that $L^2(\mathcal{N},\tau)=\overline{sp}^{\|\cdot\|_2}(\mathcal{P}\xi_1\mathcal{A}+\ldots+\mathcal{P}\xi_v\mathcal{A})$ for some vectors $\xi_1,\ldots,\xi_v\in L^2(\mathcal{N},\tau)$. Then*

$$\delta_0(x_1,\ldots,x_m)\le 2r+2v+4\,. \tag{33}$$

L. Ge and S. Popa proved ([GePo]) that if $\mathcal{Q}$ is a finite von Neumann algebra with no atoms and with a faithful normal trace $\tau:\mathcal{Q}\to\mathbb{C}$ and if, moreover, $\alpha:\Gamma\to\mathrm{Aut}(\mathcal{Q})$ is a trace-preserving properly outer action of a countable discrete group Γ on $\mathcal{Q}$, then there exist an abelian subalgebra $\mathcal{A}\subset\mathcal{Q}$ and $\xi\in L^2(\mathcal{Q}\times_\alpha\Gamma,\tau)$ such that $L^2(\mathcal{Q}\times_\alpha\Gamma,\tau)=\overline{\mathrm{sp}}^{\|\cdot\|_2}\mathcal{Q}\xi\mathcal{A}$. In the same vein, one has the following Lemma:

Lemma 4.3. *Let $\mathcal{Q}=\mathcal{Q}_1\vee\mathcal{Q}_2\simeq\mathcal{Q}_1\otimes\mathcal{Q}_2$ be a non-prime subfactor of a II_1-factor $\mathcal{N}$. If $\mathcal{Q}$ is regular in $\mathcal{N}$, then there exist diffuse abelian subalgebras $\mathcal{A}_1\subset\mathcal{Q}_1$, $\mathcal{A}_2\subset\mathcal{Q}_2$ and an abelian subalgebra $\mathcal{A}_3\subset\mathcal{Q}'\cap\mathcal{N}$ such that the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is cyclic i.e., ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}=\overline{sp}^{\|\cdot\|_2}\mathcal{P}\xi\mathcal{A}$ for some $\xi\in L^2(\mathcal{N},\tau)$, where $\mathcal{P}=\mathcal{Q}\vee(\mathcal{Q}'\cap\mathcal{N})$ and $\mathcal{A}=\mathcal{A}_1\vee\mathcal{A}_2\vee\mathcal{A}_3$.*

Proof. Let Γ be a countable group of unitaries in $N_{\mathcal{N}}(\mathcal{Q})$ such that $\overline{\mathrm{sp}}^{||\cdot||_2}\Gamma = L^2(\mathcal{N},\tau)$. Note that $\mathcal{P} \simeq \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes (\mathcal{Q}' \cap \mathcal{N})$, $N_{\mathcal{N}}(\mathcal{Q}) \subset N_{\mathcal{N}}(\mathcal{P})$ and $\mathcal{P}' \cap \mathcal{N} \subset \mathcal{P}$. Use §2 in [Po1] to conclude that there exist maximal abelian subalgebras $\mathcal{A}_1 \subset \mathcal{Q}_1$, $\mathcal{A}_2 \subset \mathcal{Q}_2$, $\mathcal{A}_3 \subset \mathcal{Q}' \cap \mathcal{N}$ with the property that for every finite subset $W \subset \mathrm{sp}\Gamma$ (=linear span of Γ) and every $\epsilon > 0$ there exists a finite partition of the identity with projections $(p_i)_{i\in I} \subset \mathcal{A} = \mathcal{A}_1 \vee \mathcal{A}_2 \vee \mathcal{A}_3$ such that $||\sum_{i\in I} p_i w p_i - E_{\mathcal{A}}(w)||_2 < \epsilon$ $\forall w \in W$, where $E_{\mathcal{A}} : \mathcal{N} \to \mathcal{A}$ is the conditional expectation onto $\mathcal{A}$. Pick a vector $\xi \in L^2(\mathcal{N},\tau)$ such that $(\xi, u) \neq 0$ $\forall u \in \Gamma$. In fact, one can assume that $\xi = v \in \mathcal{U}(\mathcal{N})$ since the set of unitaries $v \in \mathcal{U}(\mathcal{N})$ such that $(v,u) = \tau(vu^*) \neq 0$ $\forall u \in \Gamma$ is a G_δ-dense subset in $\mathcal{U}(\mathcal{N})$ ([GePo]). Let $(w_n)_{n\geq 1} \subset \mathrm{sp}\Gamma$ be an orthonormal basis of $L^2(\mathcal{N},\tau)$ and write $\xi = \sum_{n\geq 1} \alpha_n w_n$, $\xi_m = \sum_{n\geq 1}^m \alpha_n w_n$, where $\alpha_n = (\xi, w_n) \in \mathbb{C}$ $\forall n \geq 1$. Given $\delta > 0$, there exists $m \geq 1$ such that $||\xi - \xi_m||_2 < \delta$. For $u \in \Gamma$ and $\epsilon > 0$ let $(p_i)_{i\in I}$ be a finite partition of the identity with projections from $\mathcal{A}$ such that $||\sum_{i\in I} u p_i u^* w_n p_i - u E_{\mathcal{A}}(u^* w_n)||_2 < \epsilon$ $\forall 1 \leq n \leq m$. Note that

$$\begin{aligned}(34)\quad \Big|\Big|\sum_{i\in I} u p_i u^* \xi p_i - u E_{\mathcal{A}}(u^*\xi)\Big|\Big|_2 &\leq \Big|\Big|\sum_{i\in I} u p_i u^* (\xi - \xi_m) p_i\Big|\Big|_2 \\ &+ \Big|\Big|\sum_{i\in I} u p_i u^* \xi_m p_i - u E_{\mathcal{A}}(u^*\xi_m)\Big|\Big|_2 + \Big|\Big| u E_{\mathcal{A}}(u^*(\xi - \xi_m))\Big|\Big|_2 \\ &< \delta + \sum_{n=1}^m |\alpha_n|\epsilon + \delta,\end{aligned}$$

hence $u \in \overline{\mathrm{sp}}^{||\cdot||_2}\mathcal{P}\xi\mathcal{A}$ since $\epsilon, \delta > 0$ can be chosen arbitrarily small, $u p_i u^* \in \mathcal{P}$, $p_i \in \mathcal{A}$ $\forall i \in I$, $\mathcal{A}$ has no atoms, and $\tau(\xi u^*) \neq 0$. □

Theorem 4.2. *Let $\mathcal{N}$ be a subfactor of finite index in the interpolated free group factor $\mathcal{M} = \mathcal{L}(\mathbb{F}_t)$ $(1 < t \leq \infty)$ and let also r denote the integer part of the index. The following statements are true:*
i) $\mathcal{N}$ does not have regular non-prime subfactors;
ii) the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is not finitely generated if $\mathcal{A}$ is an abelian subalgebra of $\mathcal{N}$ and $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ is a subalgebra of $\mathcal{N}$ such that $\mathcal{P}_1 = \mathcal{R}_1' \cap \mathcal{P}$ and $\mathcal{P}_2 = \mathcal{R}_2' \cap \mathcal{P}$ are both diffuse, $\mathcal{R}_1$, $\mathcal{R}_2$ are mutually commuting hyperfinite subfactors of $\mathcal{P}$, and $\mathcal{R}_1 \cap \mathcal{A}$, $\mathcal{R}_2 \cap \mathcal{A}$ have projections of arbitrarily small trace;
iii) the correspondence ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}}$ is not $\mathcal{M}$-weakly contained in any finitely generated correspondence ${}_{\mathcal{P}}H_{\mathcal{A}}$ if $2r + 2v + 4 < t \leq \infty$, $\mathcal{A}$ is an abelian subalgebra of $\mathcal{N}$ and $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$ is a subalgebra of $\mathcal{N}$ such that $\mathcal{P}_1 = \mathcal{R}_1' \cap \mathcal{P}$ and $\mathcal{P}_2 = \mathcal{R}_2' \cap \mathcal{P}$ are both diffuse and $\mathcal{R}_1$, $\mathcal{R}_2$ are

mutually commuting hyperfinite subfactors of $\mathcal{P}$;
iv) $\mathcal{N}$ does not have regular diffuse hyperfinite $*$-subalgebras (DHSA).

Proof. i) Assume that $\mathcal{N}$ has a regular nonprime subfactor $\mathcal{Q} = \mathcal{Q}_1 \vee \mathcal{Q}_2 \simeq \mathcal{Q}_1 \otimes \mathcal{Q}_2$ and denote $\mathcal{P} = \mathcal{Q} \vee (\mathcal{Q}' \cap \mathcal{N})$. By Lemma 4.3, there exist diffuse abelian subalgebras $\mathcal{A}_1 \subset \mathcal{Q}_1$, $\mathcal{A}_2 \subset \mathcal{Q}_2$, an abelian subalgebra $\mathcal{A}_3 \subset \mathcal{Q}' \cap \mathcal{N}$, and a vector $\xi \in L^2(\mathcal{N}, \tau)$ such that $L^2(\mathcal{N}, \tau) = \overline{\mathrm{sp}}^{\|\cdot\|_2} \mathcal{P}\xi\mathcal{A}$, where $\mathcal{A} = \mathcal{A}_1 \vee \mathcal{A}_2 \vee \mathcal{A}_3$. We consider first the case $\mathcal{M} = \mathcal{L}(\mathbb{F}_t)$ with $1 < t < \infty$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are diffuse, for any $k \geq 1$, there exist projections $p_1 \in \mathcal{A}_1$, $p_2 \in \mathcal{A}_2$ such that $\tau(p_1) = \tau(p_2) = \frac{1}{k}$. Let $(e_{ij})_{1 \leq i,j \leq k} \subset \mathcal{Q}_1$, $(f_{ls})_{1 \leq l,s \leq k} \subset \mathcal{Q}_2$ be two matrix units such that $e_{11} = p_1$, $f_{11} = p_2$ and denote $p = p_1 p_2$. Use Lemma 4.2 to conclude

$$L^2(\mathcal{N}_p, \tau_p) = pL^2(\mathcal{N}, \tau)p = \overline{\mathrm{sp}}^{\|\cdot\|_2} \sum_{1 \leq i,l \leq k} \mathcal{P}_p \xi_{il} \mathcal{A}_p \,,$$

where $\xi_{il} = e_{1i} f_{1l} \xi p \ \forall 1 \leq i, l \leq k$. Let $\mathcal{R}_1 \subset (\mathcal{Q}_2)_p$, $\mathcal{R}_2 \subset (\mathcal{Q}_1)_p$ be hyperfinite subfactors and denote $\mathcal{P}_1 = \mathcal{R}_1' \cap \mathcal{P}_p$ and $\mathcal{P}_2 = \mathcal{R}_2' \cap \mathcal{P}_p$. If the integer part of $[\mathcal{M} : \mathcal{N}]$ is equal to r, then the integer part of $[\mathcal{M}_p : \mathcal{N}_p]$ is also equal to r and the estimate of free entropy dimension from Corollary 4.1 implies

$$\delta_0(x_1, \ldots, x_m) \leq 2r + 2k^2 + 4 \tag{35}$$

for any system $(x_1, \ldots, x_m)$ of self-adjoint generators of $\mathcal{M}_p$. On the other hand, by the compression formula ([Dy1], [Ră]),

$$\mathcal{M}_p \simeq \mathcal{L}\left(\mathbb{F}_{1+(t-1)\tau(p)^{-2}}\right) = \mathcal{L}\left(\mathbb{F}_{1+(t-1)k^4}\right).$$

Moreover ([Vo2], [Vo3]), $\mathcal{L}\left(\mathbb{F}_{1+(t-1)k^4}\right)$ has a system of generators $(x_1, \ldots, x_m)$ with $\delta_0(x_1, \ldots, x_m) = 1 + (t-1)k^4$, hence the inequality (35) implies $1 + (t-1)k^4 \leq 2r + 2k^2 + 4$ which is of course impossible if k is sufficiently large.

Let us consider now the case $\mathcal{M} = \mathcal{L}(\mathbb{F}_\infty)$, when ([Vo1]) $\mathcal{M}$ is generated by an infinite semicircular system $(x_i)_{i \geq 1}$. With the estimate of free entropy (14) we conclude that there exist elements $\left(m_j^{(e)}\right)_{j,e}$, $(p^{(t)})_t$, $(q^{(s)})_s$, $(z_k)_k$, $(p_q)_q$ (as stated in the proof of Theorem 4.1) such that

$$\chi\left((x_i)_{1 \leq i \leq m} : \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q\right) < \chi\left((x_i)_{1 \leq i \leq m}\right).$$

Let $\mathcal{M}_n = \{x_1, \ldots, x_n\}''$ and $E_n : \mathcal{M} \to \mathcal{M}_n$ be the conditional expectation onto $\mathcal{M}_n$. Since

$$\begin{aligned}(36)\qquad &\Bigg((x_i)_{1\le i\le m}, \left(E_n\left(m_j^{(e)}\right)\right)_{j,e}, \left(E_n\left(p^{(t)}\right)\right)_t, \left(E_n\left(q^{(s)}\right)\right)_s, \\ &\qquad (E_n(z_k))_k, (E_n(p_q))_q \Bigg)_{n\ge 1}\end{aligned}$$

converges in distribution as $n \to \infty$ to

$$\left((x_i)_{1\le i\le m}, \left(m_j^{(e)}\right)_{j,e}, (p^{(t)})_t, (q^{(s)})_s, (z_k)_k, (p_q)_q \right)$$

there exists an integer $n > m$ such that

$$\begin{aligned}(37)\qquad &\chi\Bigg((x_i)_{1\le i\le m} : \left(E_n\left(m_j^{(e)}\right)\right)_{j,e}, \left(E_n\left(p^{(t)}\right)\right)_t, \left(E_n\left(q^{(s)}\right)\right)_s, \\ &\qquad (E_n(z_k))_k, (E_n(p_q))_q \Bigg) < \chi\left((x_i)_{1\le i\le m}\right),\end{aligned}$$

hence

$$\begin{aligned}(38)\qquad \chi\left((x_i)_{1\le i\le n}\right) &= \chi\Bigg((x_i)_{1\le i\le n} : \left(E_n\left(m_j^{(e)}\right)\right)_{j,e}, \left(E_n\left(p^{(t)}\right)\right)_t, \\ &\quad \left(E_n\left(q^{(s)}\right)\right)_s, (E_n(z_k))_k, (E_n(p_q))_q \Bigg) \\ &\le \chi\Bigg((x_i)_{1\le i\le m} : \left(E_n\left(m_j^{(e)}\right)\right)_{j,e}, \left(E_n\left(p^{(t)}\right)\right)_t, \\ &\quad \left(E_n\left(q^{(s)}\right)\right)_s, (E_n(z_k))_k, (E_n(p_q))_q \Bigg) \\ &\quad + \chi\left(x_{m+1}, \ldots, x_n\right) < \chi\left(x_1, \ldots, x_m\right) + \chi\left(x_{m+1}, \ldots, x_n\right),\end{aligned}$$

contradiction.

ii) The statement is a direct consequence of the free entropy dimension estimate from Corollary 4.1 if $2r+2v+4 < t < \infty$. If $1 < t \le 2r+2v+4$, first cut down by a projection $p = p_1p_2$ with $p_1 \in \mathcal{R}_1 \cap \mathcal{A}$, $p_2 \in \mathcal{R}_1 \cap \mathcal{A}$ of sufficiently small trace. Note that this increases δ_0 as in the proof of i) and use then Lemma 4.2 and the free entropy dimension estimate from Corollary 4.1. The case $t = \infty$ can be treated as in the proof of i).

iii) The case $2r + 2v + 4 < t < \infty$ is consequence of the estimate of free entropy dimension from Theorem 4.1. The case $t = \infty$ can be also treated as in the proof of i).

iv) Let $\mathcal{Q} \subset \mathcal{N}$ be a regular DHSA of $\mathcal{N}$. As in the proof of Lemma 4.3, conclude that there exist a diffuse abelian subalgebra $\mathcal{A}_1 \subset \mathcal{Q}$ and an

abelian subalgebra $\mathcal{A}_3 \subset \mathcal{Q}' \cap \mathcal{N}$ such that ${}_{\mathcal{P}}L^2(\mathcal{N},\tau)_{\mathcal{A}} = \overline{\mathrm{sp}}^{||\cdot||_2}\mathcal{P}\xi\mathcal{A}$ for some $\xi \in L^2(\mathcal{N},\tau)$, where $\mathcal{P} = \mathcal{Q} \vee (\mathcal{Q}' \cap \mathcal{N})$ and $\mathcal{A} = \mathcal{A}_1 \vee \mathcal{A}_3$. Since $\mathcal{Q}$ is a DHSA of $\mathcal{N}$, this implies (with the notations from Theorem 4.1) that for any $\epsilon > 0$ there exist mutually commuting hyperfinite subfactors $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{N}$ (depending on ϵ) such that

$$\mathrm{dist}_{||\cdot||_2}\left(E_{\mathcal{N}}(m_j^* x_i), \mathrm{sp}\mathcal{R}_1\mathcal{R}_2\xi\mathcal{A}\right) < \epsilon\, \forall 1 \leq i \leq m\, \forall 1 \leq j \leq r+1.$$

As in the proof of Theorem 4.1, one obtains the free entropy dimension estimate $\delta_0(x_1, \ldots, x_m) \leq 2r+6$ and then iv) follows from this estimate in a fashion similar to the proof of i). □

Corollary 4.2. *The subfactors $\mathcal{N}$ of finite index in the interpolated free group factors $\mathcal{L}(\mathbb{F}_t)$ $(1 < t \leq \infty)$ are not crossed products of nonprime subfactors or diffuse hyperfinite subalgebras by properly outer actions of countable discrete groups.*

Proof. Let $\mathcal{Q} \subset \mathcal{N}$ be either a nonprime subfactor or a diffuse hyperfinite subalgebra. Recall that if Γ is a countable discrete group and if $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{Q})$ is a properly outer action of Γ on $\mathcal{Q}$ such that $\mathcal{N} \simeq \mathcal{Q} \times_\alpha \Gamma$ then $\mathcal{Q}$ is regular in $\mathcal{N}$. Use then i) and iv) from Theorem 4.2. □

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